

Large stars with few colors

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Abstract

A recent question in generalized Ramsey theory is that for fixed positive integers $s \leq t$, at least how many vertices can be covered by the vertices of no more than s monochromatic members of the family \mathcal{F} in every edge coloring of K_n with t colors. This is related to an old problem of Chung and Liu: for graph G and integers $1 \leq s < t$ what is the smallest positive integer $n = R_{s,t}(G)$ such that every coloring of the edges of K_n with t colors contains a copy of G with at most s colors. We answer this question when G is a star and s is either $t - 1$ or $t - 2$ generalizing the well-known result of Burr and Roberts.

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1 Introduction

Ramsey theory is an area of combinatorics which uses techniques from many branches of mathematics and is currently among the most active areas in combinatorics. Let G_1, \dots, G_c be graphs. The *Ramsey number* denoted by $r(G_1, \dots, G_c)$ is defined to be the least number p such that if the edges of the complete graph K_p are arbitrarily colored with c colors, then for some i the spanning subgraph whose edges are colored with the i -th color contains G_i . More information about the Ramsey numbers of known graphs can be found in the survey [9].

There are various types of Ramsey numbers that are important in the study of classical Ramsey numbers and also hypergraph Ramsey numbers. A question recently proposed by Gyárfás et al. in [5]; for fixed positive integers $s \leq t$, at least how many vertices can be covered by the vertices of no more

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than s monochromatic members of the family \mathcal{F} in every edge coloring of K_n with t colors. This is related to an old problem of Chung and Liu [3]: for a given graph G and for fixed $1 \leq s < t$, find the smallest $n = R_{s,t}(G)$ such that in every t -coloring of the edges of K_n there is a copy of G colored with at most s colors. Note that for $s = 1$ this is the same Ramsey number $r_t(G)$. Several problems and interesting conjectures was presented in [5]. A basic problem here is to find the largest s -colored element of \mathcal{F} that can be found in every t -coloring of K_n . The answer for matchings when $s = t - 1$ was given in [5]; every t -coloring of K_n contains a $(t - 1)$ -colored matching of size k provided that $n \geq 2k + \lceil \frac{k-1}{2^{t-1}-1} \rceil$. Note that for $t = 2, 3, 4$, we can guarantee the existence of a $(t - 1)$ -colored path on $2k$ vertices instead of a matching of size k . This was proved in [4], [8], and [7], respectively. For complete graphs the problem was partially answered in [3] and [6]. Naturally, for these graphs the answer is very few known and there are many open problems. For stars, when $s = 1$ it is the well-know result of Burr and Roberts [1], and when $s = t - 1 = 2$ it was determined in [2].

In this paper we find the value of $R_{s,t}(G)$ when G is a star and s is either $t - 1$ or $t - 2$. This will generalize the results of [1] and [2]. The paper is organized as follows. In section 2, we give the upper bound and lower bound of $R_{t-l,t}(K_{1,n})$ for given integer $l \geq 1$. In sections 3 and 4, we determine the values of $R_{t-1,t}(K_{1,n})$ and $R_{t-2,t}(K_{1,n})$, respectively. As usual, we only concerned with undirected simple finite graphs and for the vertex v of G the set of edges adjacent to v in G is denoted by $E_G(v)$.

2 Some bounds

In this section, we find some bounds for $R_{t-l,t}(K_{1,n})$. The *Turán number* $ex(H, p)$ is the maximum number of edges in a graph on p vertices which is H -free, i.e. it does not have H as a subgraph. It is easily seen that $ex(K_{1,n}, p) \leq \frac{p(n-1)}{2}$. This fact yields an upper bound for $R_{t-l,t}(K_{1,n})$ as we see in the following theorem.

Theorem 2.1 *Suppose that $t' = \lfloor t/l \rfloor$, then $R_{t-l,t}(K_{1,n}) \leq p$ for $p > \frac{t'n-1}{t'-1}$.*

Proof. Consider an edge coloring of K_p with t colors. Divide these t colors into $t' = \lfloor t/l \rfloor$ classes each of which contains l colors except the last one which may contains more colors. There exist l colors with at most $\lceil \frac{1}{t'} \binom{p}{2} \rceil$ edges. Thus the remaining $t - l$ colors appear on at least $\binom{p}{2} - \lceil \frac{1}{t'} \binom{p}{2} \rceil$ edges and the existence of $K_{1,n}$ with these $t - l$ colors is guaranteed if

$$\binom{p}{2} - \left\lceil \frac{1}{t'} \binom{p}{2} \right\rceil > \frac{p(n-1)}{2}.$$

So if $p > \frac{t'n-1}{t'-1}$, the above inequality is fulfilled and there exists a $K_{1,n}$ with at most $t - l$ colors. -1

The next theorem gives a lower bound for $R_{t-l,t}(K_{1,n})$.

Theorem 2.2 *Let $y = \left\lceil \frac{t(n-l+1)-l}{t-l} \right\rceil$. Then $R_{t-l,t}(K_{1,n}) > y - \epsilon$ where $\epsilon = 1$ if y is odd and $\epsilon = 0$, otherwise.*

Proof. Let $p = y - \epsilon$. It is sufficient to give an edge coloring of K_p such that the set of colors appear on the edges of every $K_{1,n}$ contains at least $t - l + 1$ colors. By Vizing's theorem, there exists a proper edge coloring of K_p with $p - 1$ colors. Let $p - 1 = qt + r$, $0 \leq r \leq t - 1$. We partition the above $p - 1$ colors into t classes each of which contains $q = \left\lfloor \frac{p-1}{t} \right\rfloor$ colors except the last one which may contains $(p - 1) - q(t - 1)$ colors. Every $K_{1,n}$ contains at least $t - l + 1$ colors if

$$n > (t - l - 1)q + p - 1 - (t - 1)q = (p - 1) - lq.$$

The above inequality holds if $\frac{p-1}{t} \geq \frac{p-n-1}{l} + 1$ or equivalently, $p \leq \frac{t(n-l+1)-l}{t-l}$ as asserted in Theorem 2.2. So there is no $K_{1,n}$ with at most $t - l$ colors, that is, $R_{t-l,t}(K_{1,n}) > p$. \dashv

Combining Theorems 2.1 and 2.2, we have an approximation of the value of $R_{t-l,t}(K_{1,n})$. For the small values of l this approximation is closer to the exact value. In particular, for $l = 1, 2$, we have the following corollaries.

Corollary 2.3 *Let $x = \left\lceil \frac{nt-1}{t-1} \right\rceil$. Then*

$$x \leq R_{t-1,t}(K_{1,n}) \leq x + 1.$$

In particular, when x is even, then $R_{t-1,t}(K_{1,n}) = x + 1$.

Corollary 2.4 *Let $t \geq 4$, $t' = \lfloor t/2 \rfloor$ and $x = \left\lceil \frac{nt'-1}{t'-1} \right\rceil$. Then*

$$x - 2 \leq R_{t-2,t}(K_{1,n}) \leq x + 1.$$

In particular, when $\left\lceil \frac{t(n-1)-2}{t-2} \right\rceil$ is even, then $x - 1 \leq R_{t-2,t}(K_{1,n}) \leq x + 1$.

Remark. Let v_1, \dots, v_x be vertices of K_x , where x is odd. Eliminating v_x , there exists corresponding matching M_{v_x} containing $(x - 1)/2$ parallel edges $v_1v_{x-1}, v_2v_{x-2}, \dots, v_{(x-1)/2}v_{(x+1)/2}$. Order these edges as above. Similarly, for each vertex v_i , $1 \leq i \leq x - 1$, there exists the matching M_{v_i} containing $(x - 1)/2$ ordered edges. These matchings are used to construct certain edge colorings of K_x , for example in the proof of following key lemmas.

Lemma 2.5 *Suppose that q is even and $x - 1 = tq$. There exists an edge coloring of K_x with t colors such that the set of all neighbors of every vertex contains q edges of any color.*

Proof. Partition the vertices of K_x as a single vertex v_x plus q classes T_1, \dots, T_q where T_i contains t vertices say v_{i1}, \dots, v_{it} . Set $q/2$ classes $T_1, \dots, T_{q/2}$ on one side of v_x and $q/2$ classes $T_{q/2+1}, \dots, T_q$ on the other side of v_x (see (a) of figure 1). For each vertex v_{ij} , $1 \leq j \leq t$ and $1 \leq i \leq q$, color all $(x-1)/2$ parallel edges in $M_{v_{ij}}$ with color j . Moreover, for vertex v_x , color the edge $v_{ij}v_{(q+1-i)j}$ in M_{v_x} with j . The result is a coloring of K_x with the property that the set of all neighbors of every vertex contains q edges of any color, as desired. \dashv

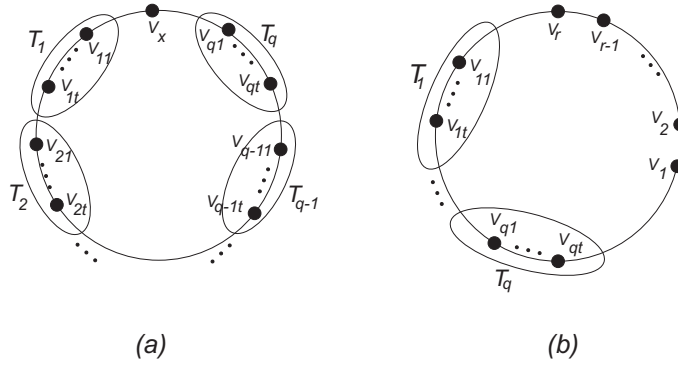


Figure 1: Partitions of the vertices of K_x

Lemma 2.6 *Suppose that $x = tq + r$ is odd and $2 \leq r \leq t - 1$. There exists an edge coloring of K_x with t colors such that the set of all neighbors of every vertex contains at least q edges of any color.*

Proof. Partition the vertices of K_x as v_1, v_2, \dots, v_r plus q classes T_1, \dots, T_q where T_i , $1 \leq i \leq q$, contains t vertices say v_{i1}, \dots, v_{it} (see (b) of figure 1). For each vertex v_{ij} color all $(x-1)/2$ parallel edges in $M_{v_{ij}}$ with color j . Moreover, for vertex v_r (also v_1) color the parallel edges in M_{v_r} (also in M_{v_1}) with $1, 2, \dots, t$ alternatively (also $t, t-1, \dots, 1$ alternatively). Color the remaining edges, i.e. parallel edges corresponding to v_2, \dots, v_{r-1} arbitrarily. The result is a coloring of edges of K_x with the property that for any vertex, each color appears on at least q edges, as desired. \dashv

3 The value of $R_{t-1,t}(K_{1,n})$

In this section, using Corollary 2.3, we determine the exact value of $R_{t-1,t}(K_{1,n})$.

Theorem 3.1 *Suppose that $x = \left\lfloor \frac{nt-1}{t-1} \right\rfloor$ and $q = \left\lfloor \frac{x}{t} \right\rfloor$. Then*

$$R_{t-1,t}(K_{1,n}) = \begin{cases} x & \text{if } x = tq + 1 \text{ for } x, q \text{ odd,} \\ x + 1 & \text{otherwise.} \end{cases}$$

Proof. First note that since $x = \lfloor \frac{nt-1}{t-1} \rfloor$, then $\frac{nt-1}{t-1} - 1 < x \leq \frac{nt-1}{t-1}$, or equivalently $x - x/t + 1/t \leq n < x - x/t + 1$ and so $n = x - \lfloor x/t \rfloor = x - q$. If x is even, then by Corollary 2.3, $R_{t-1,t}(K_{1,n}) = x + 1$. So we may assume that x is odd. We consider three cases as follows.

Case 1. $x = tq + 1$, where q is odd.

Consider an edge coloring of K_x with t colors. Suppose first that any color appears on q edges adjacent to every vertex. Consider a color c , then the subgraph induced by the edges with color c is q -regular and so the sum of degrees of its vertices is equal to the odd number xq , a contradiction. Thus there exist a vertex v and a color c with the property that c appears on at most $q - 1$ edges adjacent to v . Then there are at least $x - 1 - (q - 1) = x - q = n$ edges adjacent to v such that c does not appear on these edges. Hence there exists a subgraph $K_{1,n}$ without color c in K_x , i.e. $R_{t-1,t}(K_{1,n}) \leq x$ and so by Corollary 2.3, $R_{t-1,t}(K_{1,n}) = x$.

Case 2. $x = tq + 1$, where q is even.

In the coloring of K_x given by Lemma 2.5, every $K_{1,n}$ contains all t colors, i.e. $R_{t-1,t}(K_{1,n}) > x$ and so by Corollary 2.3, $R_{t-1,t}(K_{1,n}) = x + 1$.

Case 3. $x = tq + r$, where $2 \leq r \leq t - 1$.

In the coloring of K_x given by Lemma 2.6, every $K_{1,n}$ contains all t colors, i.e. $R_{t-1,t}(K_{1,n}) > x$ and so by Corollary 2.3, $R_{t-1,t}(K_{1,n}) = x + 1$. \dashv

As a corollary, we have the value of standard Ramsey number $r_2(K_{1,n})$ (see [9]).

Corollary 3.2 $r_2(K_{1,n}) = 2n - \epsilon$ where $\epsilon = 1$ if n is even and $\epsilon = 0$, otherwise.

4 The value of $R_{t-2,t}(K_{1,n})$

In this section, we determine $R_{t-2,t}(K_{1,n})$. Corollary 2.4 gives a lower bound and an upper bound for $R_{t-2,t}(K_{1,n})$ for $t \geq 4$. Let us first settle the case $t = 3$. It is also a special case of multi-color Ramsey numbers for stars obtained in [1].

Lemma 4.1 *There exists an edges coloring of K_{3n-2} with 3 colors such that every vertex contains exactly $n - 1$ edges from each color.*

Proof. If $3n - 2$ is even, then Vizing's Theorem gives a proper edge coloring of K_{3n-2} with $3n - 3$ colors. Divide these $3n - 3$ colors into 3 new color classes each of which contains $n - 1$ colors to get the desired coloring of K_{3n-2} with 3 colors. Thus we may assume that $3n - 2$ is odd. Then K_{3n-2} has $3n - 2$ matchings each of which contains $(3n - 3)/2$ parallel edges. For every vertex, color the corresponding parallel edges with 1, 2 and 3 respectively to get the

desired coloring. \dashv

Theorem 4.2 *It holds $R_{1,3}(K_{1,n}) = 3n - 1$.*

Proof. Consider an arbitrary edge coloring of K_{3n-1} with 3 colors 1, 2, 3 and a vertex v . Suppose that 3 is a color with maximum number of edges adjacent to v . So two colors 1 and 2 appear on at most $2\lceil \frac{3n-2}{3} \rceil$ edges adjacent to v . It is easily seen that $3n - 2 - 2\lceil \frac{3n-2}{3} \rceil \geq n$ and so we have a $K_{1,n}$ with color 3, i.e. $R_{1,3}(K_{1,n}) \leq 3n - 1$. To prove $R_{1,3}(K_{1,n}) \geq 3n - 1$, apply Lemma 4.1. In this coloring of K_{3n-2} every $K_{1,n}$ contains at least 2 colors and so $R_{1,3}(K_{1,n}) > 3n - 2$. \dashv

For general case $t \geq 4$, we let R stands for $R_{t-2,t}(K_{1,n})$, $t' = \lfloor t/2 \rfloor$ and $x = \lfloor \frac{nt'-1}{t'-1} \rfloor$.

Lemma 4.3 *Suppose that $x - 2 = tq + r$ where $0 \leq r \leq t - 1$ and l is a natural number. Then $x - l - 2q < n$ iff $t > (2r + 4)/l$ when t is even and $t > 1 + (2q + 2r + 4)/l$, otherwise.*

Proof. Since $n = x - \lfloor \frac{x}{t'} \rfloor$, we have $x - l - 2q < n$ iff $\lfloor \frac{x}{t'} \rfloor < 2q + l$ or equivalently, $\frac{x}{t'} < 2q + l$. So $x - l - 2q < n$ iff $t > (2r + 4)/l$ when t is even and $t > 1 + (2q + 2r + 4)/l$, otherwise. \dashv

Theorem 4.4, states the necessary and sufficient conditions for R being $x + 1$.

Theorem 4.4 *Suppose that $x - 2 = tq + r$ where $0 \leq r \leq t - 1$. Then $R = x + 1$ iff one the following conditions holds.*

- (a) $r = t - 1 > 2q + 4$ and x is even.
- (b) $r = t - 1 > 2q + 4$ and x and t are odd.
- (c) $r = t - 1$, x is odd and t and $q + 1$ are even.
- (d) $r < t - 2$ and $t > 2r + 4$ is even.
- (e) $r < t - 2$ and $t > 2q + 2r + 5$ is odd.

Proof. We first suppose that x is even and consider three cases as follows.

Case 1.1. $r = t - 1$.

Note that since $x - 1 = t(q + 1)$ is odd, t can't be even. Let $t > 2q + 5$. To prove $R = x + 1$, using Corollary 2.4, it is enough to give a coloring of K_x with t colors such that every $K_{1,n}$ contains at least $t - 1$ colors. By Vizing's Theorem, there exists a proper edge coloring of K_x with $x - 1$ colors. We partition these $x - 1$ colors into t color classes each of which contains $q + 1$ colors to get a coloring of K_x with t colors. Then every $K_{1,n}$ contains at least

$t - 1$ colors iff $x - 1 - 2(q + 1) < n$ which holds by the assertion and Lemma 4.3 for $l = 3$. Now let $t \leq 2q + 5$. Suppose that an arbitrary edge coloring of K_x with t colors is given. For each vertex v , there are least two colors that appear on at most $2(q + 1)$ edges of $E_G(v)$, since $x - 1 = t(q + 1)$. Using Lemma 4.3 for $l = 3$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq x$.

Case 1.2. $r = t - 2$.

We now prove $R \neq x + 1$ by showing that $R \leq x$. Suppose that an arbitrary edge coloring of K_x with t colors is given. For each vertex v , there are two colors that appear on at most $2q + 1$ edges of $E_G(v)$, since $x - 1 = t(q + 1) - 1$. Using Lemma 4.3 for $l = 2$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, there exists a $K_{1,n}$ with at most $t - 2$ colors.

Case 1.3. $r < t - 2$.

Let either $t > 2r + 4$ be even or $t > 2q + 2r + 5$ be odd. To prove $R = x + 1$, it is enough to give a coloring of K_x with t colors such that every $K_{1,n}$ contains at least $t - 1$ colors. By Vizing's Theorem, there is a proper edge coloring of K_x with $x - 1$ colors. We partition these $x - 1$ colors into $t - r - 1$ color classes each of which contains q colors plus $r + 1$ color classes each of which contains $(q + 1)$ colors to get a coloring of K_x with t colors. Then every $K_{1,n}$ contains at least $t - 1$ colors iff $x - 1 - 2q < n$ which holds by the assertion and Lemma 4.3 for $l = 1$, that is, $R > x$.

Now suppose that either $t \leq 2r + 4$ or $t \leq 2q + 2r + 5$ is odd. Suppose that an arbitrary edge coloring of K_x is given. For each vertex v , there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $x - 1 = tq + r + 1 < t(q + 1) - 1$. Hence by the assertion and Lemma 4.3 for $l = 1$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq x$.

Now suppose that x is odd. We consider three cases as follows.

Case 2.1. $r = t - 1$.

Let either $t > 2q + 5$ be odd or both of t and $q + 1$ be even. We show that $R = x + 1$. Note that since $x - 1 = t(q + 1)$ is even, if t is odd, then $q + 1$ is even. By Lemma 2.5, there exists an edge coloring of K_x with t colors such that for each vertex v , $E_G(v)$ contains $q + 1$ edges of any color. What is left is similar to the Case 1.1. If $t \leq 2q + 5$ is odd and $q + 1$ is even, similar argument as in the Case 1.1 yields $R \leq x$. Assume that $q + 1$ is odd and hence t is even. Suppose that an arbitrary edge coloring of K_x is given. If for each vertex v , $E_G(v)$ contains $q + 1$ edges of any color, the induced subgraph on the edges with a fixed color is $(q + 1)$ -regular with x vertices, a contradiction. So there exists a vertex v and a color c such that $E_G(v)$ contains at most q edges with color c . So there are two colors that appear on at most $2q + 1$ edges of $E_G(v)$. Since $x - 1 - (2q + 1) \geq n$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq x$.

Case 2.2. $r = t - 2$.

By the same argument as the Case 1.2, we get $R \leq x$.

Case 2.3. $r < t - 2$.

Let either $t > 2r + 4$ be even or $t > 2q + 2r + 5$ be odd. By Lemma 2.6, there exists an edge coloring of K_x such that for each vertex v , $E_G(v)$ contains at least q edges of any color. What is left is similar to the Case 1.3. \dashv

Theorem 4.5, states the necessary and sufficient conditions for R being x .

Theorem 4.5 *Suppose that $x - 2 = tq + r$ where $0 \leq r \leq t - 1$. Then $R = x$ iff one the following conditions holds.*

- (a) $r = t - 1$ and x and $q + 1$ are odd.
- (b) $r < t - 2$ and $t \leq 2r + 4$ is even.
- (c) $r < t - 2$ and $q + r + 3 < t \leq 2q + 2r + 5$ is odd.

Proof. Let $p = x - 1$, then $p - 1 = tq + r$. We first suppose that p is even and consider three cases as follows.

Case 1.1. $r = t - 1$.

Let t be even and $q + 1$ be odd. By Theorem 4.4, $R \leq x$. By Vizing's Theorem there exists a proper edge coloring of K_p with $p - 1$ colors. We partition these $p - 1$ colors into $t - 1$ classes each of which contains $q + 1$ colors plus a class which contains q colors to get a coloring of K_p with t colors. Then every $K_{1,n}$ contains at least $t - 1$ colors iff $p - 1 - (2q + 1) < n$ which holds by the assertion and Lemma 4.3 for $l = 3$, that is, $R > p = x - 1$ and so $R = x$.

If both of t and $q + 1$ are even then $R > x$ by Theorem 4.4. Note that the case when both of t and $q + 1$ are odd is impossible, since $p = t(q + 1)$ is even. Assume that t is odd and $q + 1$ is even. If $t > 2q + 5$, then $R \neq x$ by Theorem 4.4. Let $t \leq 2q + 5$ be odd and $q + 1$ be even. Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v , there are two colors that appear on at most $2q + 1$ edges of $E_G(v)$, since $p - 1 = t(q + 1) - 1$. Hence by Lemma 4.3 for $l = 3$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq p = x - 1$.

Case 1.2. $r = t - 2$.

Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v , there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $p - 1 = t(q + 1) - 2$. Hence by Lemma 4.3 for $l = 2$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq p = x - 1$.

Case 1.3. $r < t - 2$.

Let either $t \leq 2r + 4$ be even or $q + r + 3 < t \leq 2q + 2r + 5$ be odd. By Theorem 4.4, $R \leq x$. By Vizing's Theorem, there exists a proper edge coloring

of K_p with $p - 1$ colors. We partition these $p - 1$ colors into $t - r$ color classes each of which contains q colors plus r color classes each of which contains $q + 1$ colors to get a coloring of K_p with t colors. Then every $K_{1,n}$ contains at least $t - 1$ colors iff $p - 1 - 2q < n$ which holds by the assertion and Lemma 4.3 for $l = 2$, that is, $R > p = x - 1$ and so $R = x$.

If either $t > 2r + 4$ is even or $t > 2q + 2r + 5$ is odd, then $R \neq x$ by Theorem 4.4. Assume that $t \leq q + r + 3$ is odd. Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v , there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $p - 1 < t(q + 1) - 2$. Hence by the assertion and Lemma 4.3 for $l = 2$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq p = x - 1$.

Now suppose that p is odd. We consider three cases as follows.

Case 2.1. $r = t - 1$.

So t and $q + 1$ are odd. If $t > 2q + 5$, then by Theorem 4.4, $R = x + 1$. Now let $t \leq 2q + 5$ be odd. Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v , there are two colors that appear on at most $2q + 1$ edges of $E_G(v)$, since $p - 1 = t(q + 1) - 1$. Hence by Lemma 4.3 for $l = 3$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq p = x - 1$.

Case 2.2. $r = t - 2$.

By the same arguments as the Case 1.2, we get $R \leq p = x - 1$.

Case 2.3. $r < t - 2$.

Let either $t \leq 2r + 4$ be even or $q + r + 3 < t \leq 2q + 2r + 5$ be odd. If $r = 0$ and t is even, then $t = 4$ and so $x = \lfloor \frac{nt'-1}{t'-1} \rfloor = 2n - 1$, which is impossible. By Lemmas 2.5 and 2.6, there exists an edge coloring of $G = K_p$ with t colors such that for each vertex v , $E_G(v)$ contains at least q edges of any color. What is left is similar to Case 1.3. \dashv

Theorem 4.6, states the necessary and sufficient conditions for R being $x - 1$.

Theorem 4.6 *Suppose that $x - 2 = tq + r$ where $0 \leq r \leq t - 1$. Then $R = x - 1$ iff one the following conditions holds.*

- (a) $r = 1$, $\frac{2q+9}{3} < t \leq q + 4$ is odd and x is even
- (b) $r = 1$, $\frac{2q+9}{3} < t \leq q + 4$ is odd and x is odd.
- (c) $1 < r < t - 2$ and $\frac{2q+2r+7}{3} < t \leq q + r + 3$ is odd.
- (d) $r = t - 2$ and either t is even or $t > \frac{2q+2r+7}{3}$ is odd.

Proof. Let $p = x - 2$, then $p = tq + r$. We first suppose that x is even and consider five cases as follows.

Case 1.1. $r = 0$.

If either t is even or $t > 2q + 5$ is odd, then $R \neq x - 1$ by Theorems 4.4 and 4.5. If $q + 3 < t \leq 2q + 5$ is odd, then $R \neq x - 1$, by Theorem 4.5. Now let $t \leq q + 3$ be odd. Note that q is even in this case. Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v , there are two colors that appear on at most $2q - 1$ edges of $E_G(v)$, since $p - 1 = x - 3 = tq - 1$. Hence by the assertion and Lemma 4.3 for $l = 2$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq p = x - 2$ and so by Corollary 2.4, $R = x - 2$.

Case 1.2. $r = 1$.

Let $\frac{2q+9}{3} < t \leq q + 4$ be odd. Since $t \leq q + 4$, by Theorems 4.4 and 4.5, $R \leq x - 1$. By Vizing's Theorem, there exists a proper edge coloring of K_p with $p - 1$ colors. We partition these $p - 1$ colors into t color classes each of which contains q colors to get a coloring of K_p with t colors. Then every $K_{1,n}$ contains at least $t - 1$ colors iff $x - 3 - 2q = p - 1 - 2q < n$ which holds by the assertion and Lemma 4.3 for $l = 3$, that is, $R > p = x - 2$ and hence $R = x - 1$. If either $t > q + 4$ is odd or t is even, then $R \neq x - 1$, by Theorems 4.4 and 4.5. Now let $t \leq \frac{2q+9}{3}$ be odd. Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v , there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $p - 1 = x - 3 = tq$. Hence by the assertion and Lemma 4.3 for $l = 3$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq p = x - 2$.

Case 1.3. $1 < r < t - 2$.

Let $\frac{2q+2r+7}{3} < t \leq q + r + 3$ be odd. Since $t \leq q + r + 3$, by Theorems 4.4 and 4.5, $R \leq x - 1$. By Vizing's Theorem, there exists a proper edge coloring of K_p with $p - 1$ colors. We partition these $p - 1$ colors into $t - r + 1$ color classes each of which contains q colors plus $r - 1$ color classes each of which contains $q + 1$ colors to get a coloring of K_p with t colors. Then every $K_{1,n}$ contains at least $t - 1$ colors iff $x - 3 - 2q = p - 1 - 2q < n$ which holds by the assertion and Lemma 4.3 for $l = 3$, that is, $R > p = x - 2$ and so $R = x - 1$.

If either $t > q + r + 3$ is odd or t is even, then $R \neq x - 1$ by Theorems 4.4 and 4.5. Now let $t \leq \frac{2q+2r+7}{3}$ be odd. Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v , there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $p - 1 = x - 3 = tq + r - 1 < t(q + 1) - 3$. Hence by the assertion and Lemma 4.3 for $l = 3$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq p = x - 2$.

Case 1.4. $r = t - 2$.

Let either t be even or $t > \frac{2q+2r+7}{3}$ be odd. By Theorems 4.4 and 4.5, $R \leq x - 1$. By Vizing's Theorem, there exists a proper edge coloring of K_p with $p - 1$ colors. We partition these $p - 1$ colors into $t - 3$ color classes each of which contains $q + 1$ colors plus 3 color classes each of which contains q colors to get a coloring of K_p with t colors. Then every $K_{1,n}$ contains at least $t - 1$

colors iff $x - 3 - 2q = p - 1 - 2q < n$ which holds by the assertion and Lemma 4.3 for $l = 3$, that is, $R > p = x - 2$. Therefore $R = x - 1$.

Now let $t \leq \frac{2q+2r+7}{3}$ be odd. Suppose that an arbitrary edge coloring of $G = K_p$ with t colors is given. For each vertex v , there are two colors that appear on at most $2q$ edges of $E_G(v)$, since $p - 1 = x - 3 = t(q + 1) - 3$. Hence by the assertion and Lemma 4.3 for $l = 3$, at least n edges of $E_G(v)$ are colored with the remaining $t - 2$ colors, that is, $R \leq p = x - 2$.

Case 1.5. $r = t - 1$.

Hence t is odd. If $t > 2q + 5$, then by Theorem 4.4, $R \neq x - 1$. Now let $t \leq 2q + 5$ be odd. Using Lemma 4.3 for $l = 3$, we have $x - 3 - 2q \geq n$ and so for each edge coloring of $G = K_p$ with t colors, n edges of $E_G(v)$ are colored with at most $t - 2$ colors, that is $R \leq p = x - 2$.

Now suppose that x is odd. We consider five cases as follows.

Case 2.1. $r = 0$.

The proof is similar to the Case 1.1. Note that when t is odd, q can't be even.

Case 2.2. $r = 1$.

Let $\frac{2q+9}{3} < t \leq q + 4$ be odd and q be even. By Lemma 2.5, there exists an edge coloring of K_p with t colors such that every vertex contains q edges of any color. What is left is similar to Case 1.2. Note that the case when both of t and $q + 1$ are odd is impossible.

Case 2.3. $1 < r < t - 2$.

Let $\frac{2q+2r+7}{3} < t \leq q + r + 3$ be odd. By Lemma 2.6, there exists an edge coloring of K_p with t colors such that every vertex contains at least q edges of any color. What is left is similar to Case 1.3.

Case 2.4. $r = t - 2$.

Let either t be even or $t > \frac{2q+2r+7}{3}$ be odd. By Lemma 2.6, there exists an edge coloring of K_p with t colors such that every vertex contains at least q edges of any color. What is left is similar to Case 1.4.

Case 2.5. $r = t - 1$.

If either t is even or $t > 2q + 5$ is odd, then $R \neq x - 1$ by Theorems 4.4 and 4.5. What is left is similar to the Case 1.5. \dashv

Corollary 4.7 $R = x - 2$ iff none of the conditions stated in Theorems 4.4, 4.5 and 4.6 holds.

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